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APPROACH TO THE INVESTIGATION OF THE OCCURRENCE OF TURBULENCE IN THE FLOW OF A VISCOUS FLUID IN CLOSED VOLUMES

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It is obviously simpler to attempt to understand the nature of turbulence by studying the processes which accompany its occurrence in the case of closed flow, where these processes reflect the inherent properties of the flows and are complicated by external random processes in a fairly small and predictable way. Such flows usually convert into turbulent flow via a number of bifurcations and a stage of dynamic chaos, namely, modes of flow with unpredictable behavior with time.

Since it is not possible at present to progress very far in describing the evolution of flows, in which the parameters change, by direct "integration" of the Navier–Stokes equations it is extremely desirable to make use of more general models. It is this that is also proposed in [1]: one should not concentrate on the specific features of actual flows but, using the ideal of universality, one should study, in general form, the types of bifurcations inherent in nonlinear dynamic systems of general form. This approach, of course, requires more serious justification since, first, the state space for the Navier–Stokes equations is infinitely dimensional and, second, it is not clear whether the properties of the dynamic systems obtained by reducing the initial equations are typical. The first steps in this direction would be to prove the existence and uniqueness of global solutions of the Navier–Stokes equations and then to prove that the attractors of these equations are of finite dimensions and obtain upper and lower estimates of their Hausdorff dimensions. There are only conventional theorems on the finite dimensionality of attractors of three-dimensional motions of a viscous liquid available at the present time [2, 3], and complete proofs only exist for the two-dimensional case [4].

A number of fundamental results have been obtained in numerous experimental and numerical investigations of the occurrence of turbulence (see, for example, [5, 6]). In particular, it has been found that chaos in closed flows begins after a small number of bifurcations and the corresponding chaotic attractors have comparatively small dimensions. It was discovered, however, that even space–time dynamic chaos is not "true" turbulence, but only one more step towards it. As the parameters increase the chaotic modes undergo further bifurcations, and the properties of the corresponding multidimensional attractors cannot be characterized by existing methods.

Moreover, as in turbulence, these modes of flow are practically indistinguishable using the standard approach. This situation gives rise to numerous problems which touch on the ways that chaos can occur and develop, the properties of the set of solutions of the Navier–Stokes equations, their bifurcations, and the possibility of classifying these solutions. Whereas the Navier–Stokes initial boundary-value problem describes the phenomenon of turbulence and the transition to it, the theory of hydrodynamic instability, which enables one, in principle, to remove the difference in the analysis of bifurcations between the partial differential equations and systems of ordinary differential equations (on the central manifold), indicates the generality of the problems that arise here for all nonlinear dissipative dynamic systems of fairly high dimensions.

Our experimental investigations of the occurrence of chaos [6, 7] in the flow between two rotating spheres (spherical Couette flow) showed that, at the initial stage (when chaos occurs) there is complete agreement with the theory of dynamic systems: there are many nonstandard transition paths. In this connection, multiparametric investigations of spherical Couette flow have been undertaken from which it follows that many unexpected bifurcations occur due to the single-parametric approach to the investigation of the evolution of particularly multiparametric systems: bifurcations of codimensions greater than 1 appear as "nontypical" and unpredictable [8]. The multiparametric approach enabled us to find regions of parameter space where there is no interaction of the bifurcational surfaces, and the scenarios by which chaos occurs are realized through typical bifurcations when only a single parameter varies.

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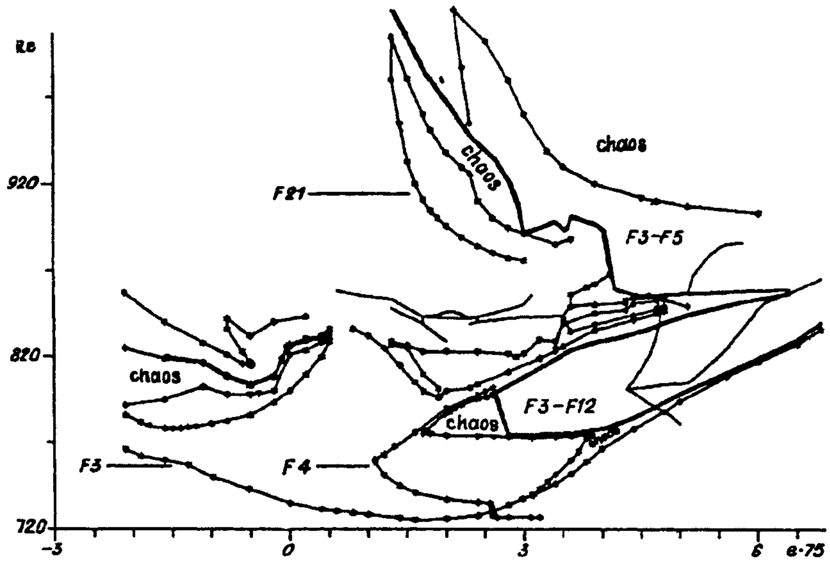


Fig. 1

To some extent these investigations remove one difficulty but reveal more fundamental ones connected with the structure of multidimensional chaotic attractors. The actual way of overcoming these difficulties remains, nevertheless, a sequential investigation of the structure and properties of multidimensional chaotic attractors from the prehistory of their occurrence. Such attempts have already been made both in experimental and theoretical investigations, but, in the final analysis, they have all encountered the difficulty that there are no effective methods of representing such objects. The point is that the trajectory, after a reasonable time, is only able to follow an insignificant part of the multidimensional set, while all the increasing numbers of independent modes of motion are more and more masked by unavoidable measurement errors.

In the most complete analysis of methods of representing the properties of chaotic attractors [9] the idea was considered that, to achieve a certain value of the dimensions of the chaotic attractor (say, 7), no reasonable volume of information suffices for a correct estimate of the dimensions. It was also suggested in [9] (and successfully realized using the example of Rayleigh–Benard convection) that, to reconstruct the phase pattern of the system, measurements of the evolution with time of not one independent variable but the maximum possible number should be used. However, in practice, this approach involves extremely high costs.

In the present paper, using the example of the investigation of one of a series of bifurcations of one of the modes of spherical Couette flow in the plane of two parameters (e and Re), an attempt is made to show that the method proposed in [10] for representing chaotic attractors in time series for a single independent variable is possible. The main advantage of this correlation method, as applied to the analysis of time series obtained experimentally, is that effective separation of the high-frequency experimental noise and the slow chaotic dynamics on the attractor is possible.

1. Conditions for Carrying Out the Experiments. We investigated the adjustment of the spherical Couette flow when the Reynolds number Te (or, what is the same thing, the Taylor number Ta) changes both for a fixed relative layer thickness $\delta = 1.006$ and a fixed eccentricity. The basic notation which will be used henceforth is as follows: the Reynolds number Re $= FR_1^2/\nu$ (F is the angular frequency of rotation of the internal sphere, R_1 is its radius and ν is the kinematic viscosity of the fluid), the relative thickness of the spherical layer $\delta = (R_2 - R_1)/R_1$ (R_2 is the radius of the outer sphere, $R_2 > R_1$), and the eccentricity $e = R/R_1$ (R is the shift of the center of the inner sphere with respect to the center of the outer sphere along the axis of rotation). Below, all the frequencies in the spectra are normalized to the quantity $F_0 = F/2\pi$, and $R_1 = 75$ mm. We made visual observations of the adjustments of the flow and we also measured the pulsations of the velocity at three points of the layer (in the region of the pole and the equator and at a latitude of 45°, approximately at the middle of the layer along the radius) or three components of the velocity at one point simultaneously. To do this we used laser Doppler velocimeters, the signals from which, after removing the constant component, were digitized and recorded in the memory of a personal computer and on a hard disk. The frequency with which each channel was interrogated was fixed and was 25 Hz.

The parameters of the equipment and the conditions under which the experiment was carried out were chosen to be such that all the main characteristics (Re, the frequency, the half-width of the spectral lines, etc.) could be kept constant to within 0.03%. The characteristic time of viscous diffusion, calculated from the thickness of the spherical layer for $\delta = 1.006$,

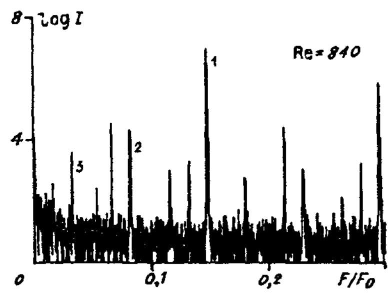


Fig. 2

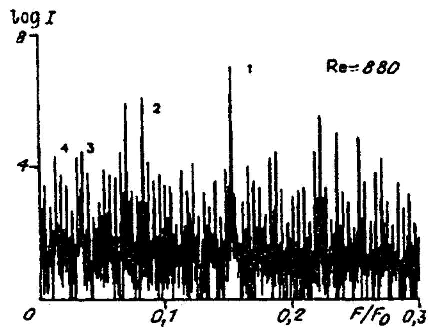


Fig. 3

was about 100 sec, and over the periphery of the layer it was about 3000 sec in all the experiments. Hence, all the measurements under so-called steady conditions were carried out after steady flow had been maintained for 40 minutes. The individual samples were as long as six hours.

2. Method of Processing the Results of Measurements. Several modes of flow were first isolated in the Re−e plane, in which a transition to chaos and further adjustment of the chaotic attractors when only Re changed occurred as a result of normal bifurcations, in every case softly and reversibly. Then, after establishment procedures, the pulsations of the flow velocity were measured at several points of the spherical layer before and after the transition. Long samples were obtained. The processing of these consisted of a Fourier and a correlation analysis and all possible combinations of these procedures, which consisted of standard methods of spectral-correlation analysis of signals and which, for regular quasi-periodic modes, enabled us to characterize the states of the system uniquely, and to judge the transition to chaos from the sharp broadening of the spectral lines and the increase in the loss of correlation. For chaotic modes the phase patterns of a system of different dimensions were reconstructed using the well-known Takens procedure (the time-delay method) in order to calculate the characteristics of the attractors of dimensions (pointwise and correlation), Lyapunov indices, etc.

Since it was obviously not possible in these experiments to encounter the simplest chaotic attractors, and the transition to chaos occurred from tori of fairly large dimensions, the standard procedures mentioned above do not enable one, in practice, to determine uniquely either the bifurcations of the chaotic attractors themselves or their dimensions or to judge from their value the readjustments that are occurring. It is under these conditions that unavoidable measurement errors have the greatest effect. We therefore attempted to use the approach proposed in [10] and its variant [11], modified somewhat.

The basic principle of the algorithm is well known: one constructs from measurement data the trajectory matrix X, made up of the vectors x_i of dimensions n, which occur as if by sequential scanning of a time series of length $N_0 \gg m$ through a time window of width n. The problem arises of determining for what direction c in n-dimensional space the sum of the squares of the projections S of these vectors will be a maximum. We obtain from the conditions for an extremum of S

$$\sum_{i=1}^{N} \sum_{k=1}^{n} x_{ij} x_{ik} c_k - S c_j = 0,\ j = 1, \ldots, n,\ N = N_0 - n + 1.$$

If we put $V = \sum_{i=1}^{N} x_{ij} x_{ik}$, we obtain an eigenvalue problem for the matrix V, proportional to the covariance matrix of the vectors x: Vc = Sc. The eigenvalues of this matrix are equal to the sum of the squares of the projections of the vectors x on the direction c, while the c themselves form an orthonormalized basis. It is assumed that the deterministic part of the time series determines the greatest eigenvalues, whereas the random noise, which makes contributions to all the eigenvalues, considerably distorts only the corresponding weakest motions, not exceeding the noise level. The projections on the first vectors, the eigenvalues of which lie above the noise level, provide estimates of the dimensions of the attractor. One of the versions of these estimates is described in [10].

Unfortunately, there are no clear criteria for separating the deterministic part of the eigenvalues from the region of transition from deterministic to noisy values. We developed a procedure to do this. We constructed the projections of the phase pattern on the first q < n eigenvectors of the covariance matrix (in this investigation q = 9 and n = 20-30), which gives a set of vectors z_i.

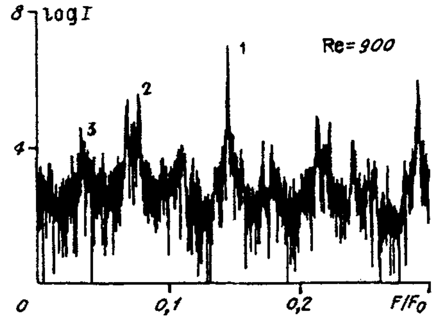


Fig. 4

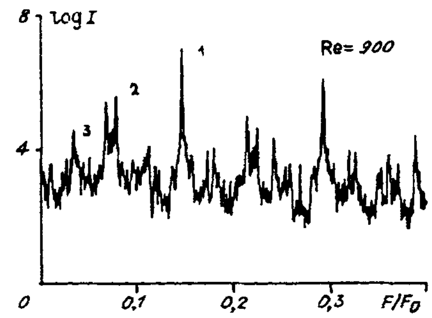
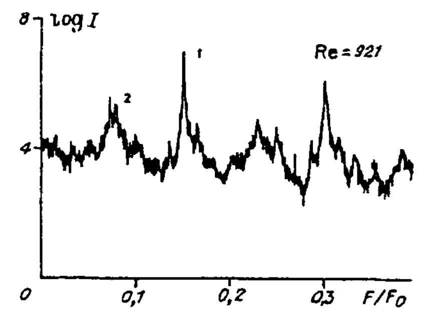


Fig. 5



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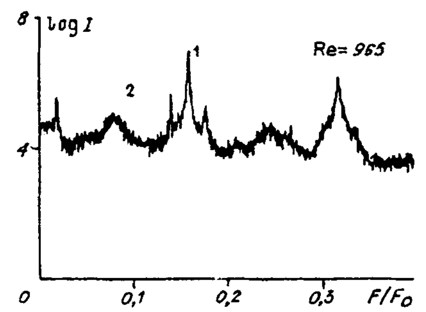

Fig. 7

We then took the set of tangential vectors $y_i = z_i - z_0$, lying in a small neighborhood E of a certain vector z_0 from a chosen region of the attractor. A local covariance basis is constructed for all y_j such that $E - \sigma < |y| < E + \sigma$. If all y_j lay strictly in tangential space, they were projected only onto the first m vectors, where m is the dimensions of the attractor, while the other projections were equal to 0 or close to the noise values. Because of the curvature of the manifold, the y_j are projected onto a large number of directions. For a certain number of directions the attractor may be fractal. However, the dependence of the length of the projections on E for these different directions is different. For the "true" projections $S \sim E^2$, due to the curvature $S \sim E^4$, and for the noise directions S does not depend on E, whereas in the fractal directions there is no regular dependence of S on E [11].

3. Results of Measurements. In Fig. 1 we show a diagram of the flow modes in the Re−e plane for one of the sequences of modes with principal frequencies F1 and F3, namely the sequences F1, F3 and F4 [8]. The bifurcational curves corresponding to the soft and reversible adjustments of the flow modes for a quasistatic change in the parameters are represented by the lines. We have used the notation F3, F4, etc., to denote the curves where motions appear with specified frequencies (for technical reasons not all the curves are so labelled), while the notation F3-F12, etc, denotes regions in which other sequences exist. The double lines represent irreversible adjustments of the flow modes. Certain regions of the chaotic motions are denoted by the word chaos. Modes of the type F3-F5 and F3-F12, although they also occur in the sheet F1, F3 and F4, in fact belong to other sheets of the general bifurcational diagram. They possess considerable hysteresis (with respect to the curves shown in Fig. 1) and may be extended backwards from the double lines to considerably smaller values of Re softly and reversibly.

In this investigation we studied one of the sequences of the mode F3-F5, to reach which we carried out the following operations. For fixed $e \approx 0.01$ and a quasistatic increase in Re we obtained initially for Re = 460 a mode with frequency F1, and then, for Re = 734, the frequency F3 appeared, the value of Re increased up to 870 and the eccentricity increased further to e = 0.073, as a result of which the mode F3-F5 occurred. One can reach this mode if, for fixed e = 0.073, after exciting F1, by changing Re quasistatistically, one obtains a chaotic mode for Re = 790 as a result of bifurcation of codimension 2, and then one sharply increases Re to values greater than 850. For fixed e, all the adjustments of this mode with frequencies F1, F3 and F5 occur softly and reversibly when only Re is varied, by increasing which other frequencies appear, chaos arises as well as a number of adjustments of the chaotic modes, and when Re is reduced these states disappear in reverse order and are reproducible.

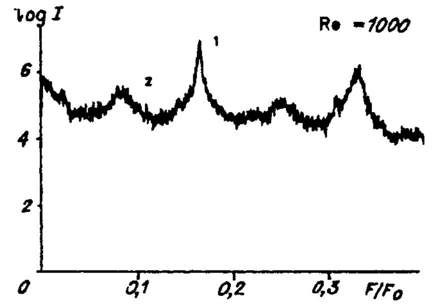

Fig. 8

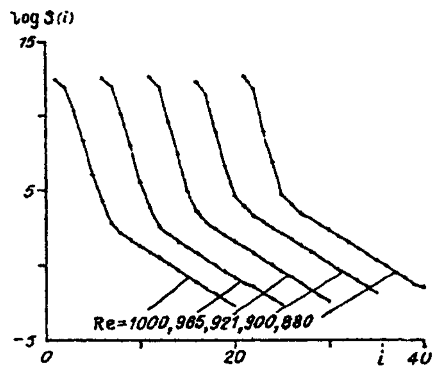

Fig. 9

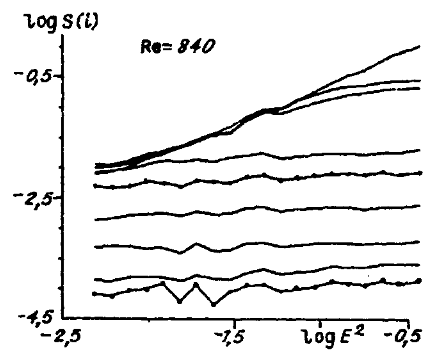

Fig. 10

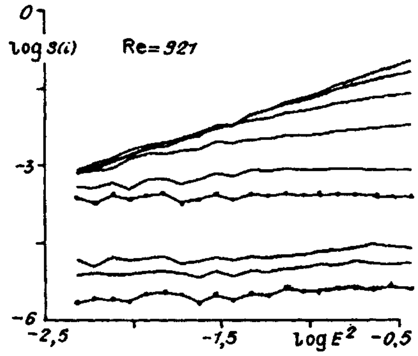

Fig. 11

We show spectrograms of some of these in Figs. 2-8. Along the abscissa axis we have plotted the frequency, normalized to the frequency of rotation of the sphere F_0, and along the ordinate axis we have plotted the logarithm of the spectral power density of the flow velocity pulsations logI. The numbers 1-4 indicate the fundamental noncommensurable frequencies of the modes representing independent motions. The spectrogram of the mode with three frequencies (Re = 840) is shown in Fig. 2 and was obtained from the data of measurements in the region of a pole where motions at frequency F1 were very small (peak 3). There are peaks in the spectrum at the fundamental frequencies, the harmonics, their combinations and broadband noise. Estimates of the dimensions of the attractor by traditional methods gave a value of 2.3 and clearly do not sufficiently take into account the contribution of the motion at frequency F1.

Bifurcation of this torus 3 occurs when Re = 875 in torus 4. The spectrogram of the new mode when Re = 880 is shown in Fig. 3. The addition of one more frequency considerably complicates the spectrum, but due to the regularity the mode can be uniquely identified. However, to obtain acceptable values of the dimension by a standard method one must increase the length of the sample. In Fig. 9 the eigenvalues of the covariance matrix S(i) for different flow modes are shown by the points, connected, for clarity, by sections of straight lines, as graphs of lnS as a function of the number i (the value of Re of the mode is indicated under each curve obtained). The four highest eigenvalues selected indicate that the dimensions of the attractor do not exceed 4, and the remaining S(i) can be regarded as noise.

When Re = 895 a chaotic attractor occurs in the system. The spectra for the new mode when Re = 900 are shown in Figs. 4 and 5. It can be seen that there is considerable broadening of all the spectral lines, and all the underlying peaks sink into a continuous background, which is reduced by three orders of magnitude compared with the regular modes. The difficulties involved in estimating the pointwise or correlation dimensions are due precisely to this nature of the evolution of the trajectory in this chaotic set. To obtain justified estimates of the spectral density distribution of this dynamic it is necessary to average many spectrograms (compare Fig. 4 (a single sample) and Fig. 5 (averaged over 13 samples)), and long samples are needed to estimate the dimensions. The eigenvalues of S(i) are shown in Fig. 9 (the curve with Re = 900). It can be seen that four

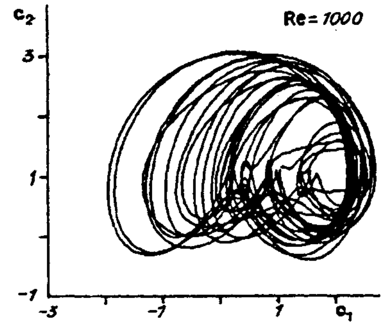

Fig. 12

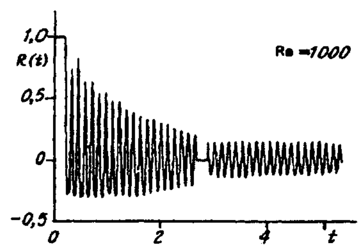

Fig. 13

principal eigenvalues are attained and hence, one of the directions of motion on torus 4 has become chaotic but the torus itself has remained.

When Re = 915 bifurcation of the chaotic attractor occurs, which it is practically impossible to detect by simply observing only the evolution of the spectrum: as Re increases there is a gradual increase in the level of the continuous background and a disappearance of the residues of the spectral lines (compare Figs. 5 and 6). Estimates of the dimensions using the correlation integral and in a pointwise manner are uncertain: as Re increases the dimensions change gradually. The eigenvalues of the covariance matrix for Re = 921 are shown in Fig. 9. Four principal eigenvalues can be distinguished but the form of the "plateau" of the initial eigenvalues, which belong to the noise, has also changed, so that the fifth eigenvalue to some extent represents mainly not the noise but the dynamics of the system. The dynamics of the system in the so-called "local covariance basis" have been analyzed in detail in [11], and the first estimates indicate that in this case torus 4 was destroyed as a result of the generation of a new motion. The results for this analysis are shown in Figs. 10 and 11. When Re = 840 (Fig. 10) only three eigenvalues in the local covariance basis manifest the required dependence on E^2 for comparatively small values of E, whereas when Re = 921 there are four such eigenvalues and a similar relationship is found for the fifth, where the oscillating form of the graphs of the latter indicates the chaotic nature of the directions on the attractor corresponding to them.

Even greater radical changes in the system occur when Re is increased further. Then occurrence of a chaotic mode, the spectrum of which is shown in Fig. 8, is obviously preceded by two bifurcations of the chaotic attractor discussed above. The topology of the attractor changes considerably – in effect, two centers appear, around which the trajectory attempts to evolve. The projection of the phase pattern of the system onto the plane of the first two eigenvectors c_1 and c_2 of the covariance basis is shown in Fig. 12. The trajectory of the system shown here is made up of 0.3% of the whole trajectory investigated, the apparent discontinuities and self-intersecting trajectory – a consequence of the two-dimensional projection of the multidimensional set. The spectra of the eigenvalue of the covariance matrix for Re = 1000 and 965 are shown in Fig. 9, and from them we can draw the preliminary conclusion that once bifurcation occurs before Re = 965, while the other occurs after this (the spectrum of the chaotic mode for Re = 965, while the other occurs after this (the spectrum of the chaotic mode for Re = 965 is shown in Fig. 7). A preliminary analysis in the local covariance basis shows that the first of these corresponds to the occurrence of a new regular motion while the second corresponds to a new chaotic motion. The attractor can be characterized by seven independent coordinates and is imbedded in a 15-dimensional Euclidean space. There are four regular directions and three chaotic directions on the attractor. Traditional methods of estimating the dimensions give, in this case, a value of 5.9, but one cannot judge the structure of the attractor from this.

In conclusion we note that by constructing a Karunen–Loev basis for the trajectory matrix it becomes possible to advance traditional methods of characterizing chaotic attractors somewhat further. Even in comparatively complex chaotic modes, the spectra of which are practically continuous and the correlations decay strongly (see the autocorrelation function for Re = 1000 in Fig. 13), it is possible to investigate the structure of the multidimensional attractors corresponding to them, and on the basis of this to classify the set of perturbulent chaotic states of hydrodynamic systems.

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